Affine Hecke Algebras: Representations

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1. Bernstein Center

$\mathscr{R} = (X^*(T), R, Y^*(T), R^\vee)$, $\mathcal{H}(\mathscr{R}) = \mathcal{H}_f \otimes \mathbb{Z}[X^*(T)]$ with cross relation

$$\theta_\lambda T_{s_\alpha} - T_{s_\alpha}\theta_{s(\lambda)} = (q-1)\frac{\theta_\lambda - \theta_{s(\lambda)}}{1-\theta_{-\alpha}}$$

Let $\lambda = s(\lambda)$ above, then adding the two equations yields

$$T_s(\theta_\lambda + \theta_{s(\lambda)}) = (\theta_\lambda + \theta_{s(\lambda)})T_s \tag{1}$$

Theorem 1 (Bernstein)
For any $\lambda \in X^*_+(T)$, define

$$z_\lambda = \sum_{w \in W_f} \theta_{w\cdot\lambda}$$

Then $\mathcal{Z}(\mathcal{H}) = \bigoplus\limits_{\lambda \in X^*_+(T)} \mathbb{Z}[q^{\pm 1}] \cdot z_\lambda \cong \mathbb{Z}[q^{\pm1}][X^*(T)]^{W_f}$.

Proof. $(1) z_\lambda \in \mathcal{Z}(\mathcal{H})$: The $\mathbb{Z}_2 = \langle s \rangle$ action on W_f partitions W_f into cosets of size 2. Thus

$$T_s z_\lambda = \sum_{w \in W_f/<s>} T_s(\theta_{w\cdot\lambda} + \theta_{sw\cdot\lambda}) \overset{\text{Eq. (1)}}{=\!=} z_\lambda T_s$$

(2) $\{z_\lambda\}$ generates $\mathcal{Z}(\mathcal{H})$. We have a map $sp : \mathcal{H} \twoheadrightarrow \mathbb{Z}[W_{ext}] = \mathbb{Z}[X^*(T) \rtimes W_f]$ given by sending $q \mapsto 1$.

Lemma 1. *Let $G \subseteq R$ faithfully where R is an integral domain. Then $\mathcal{Z}(R \rtimes G) = R^G$.*

Proof. $\underline{R^G \subseteq \mathcal{Z}(R \rtimes G)}$ $r \in R^G$ commutes with R, and note $gr = g(r)g = rg$ and thus commutes with G as well.
$\underline{\mathcal{Z}(R \rtimes G) \hookrightarrow R^G}$ Let $r \in R$, and suppose $z \in \mathcal{Z}(\mathcal{H})$, write $z = \sum\limits_{g\in G} z_g \otimes g$ where $z_g \in R$. Then

$$\sum_{g \in G} rz_g \otimes g = rz = zr = \sum_{g\in G} z_g \otimes gr = \sum_{g \in G} z_g g(r) \otimes g \quad \forall r \in R$$

Since $\{g\}_{g\in R}$ is an $R-$basis for $R \rtimes G$ and R is an integral domain it follows that

$$z_g r = z_g g(r) \implies r = g(r) \forall r \in R$$

But $G \curvearrowright R$ faithfully and thus only z_1 is nonzero and thus $z = z_1 \in R$. But now $zh = hz = h(z)h \implies z = h(z) \forall h \in G$ and thus $z \in R^G$ as desired. ■

Let $R = \mathbb{Z}[X^*(T)]$ and $G = W_f$, then $G \subseteq R$ faithfully as the only weight in the intersection of all the hyperplanes is 0. It follows that

$$\mathcal{Z}(\mathbb{Z}[W_{ext}]) = \mathbb{Z}[X^*(T)]^{W_f} = \bigoplus_{\lambda \in X^*_+(T)} \mathbb{Z} \cdot c_\lambda, \qquad c_\lambda = \sum_{w \in W_f} [w \cdot \lambda]$$

Note $\mathfrak{m} = \ker sp = (1-q)\mathcal{H}$ is a prime ideal. Since sp is surjective it restricts $sp : \mathcal{Z}(\mathcal{H}) \twoheadrightarrow \mathcal{Z}(\mathbb{Z}[W_{ext}])$ and we have the SES

$$0 \to \mathfrak{m}\mathcal{Z}(\mathcal{H}) \to \mathcal{Z}(\mathcal{H}) \xrightarrow{sp} \mathbb{Z}[X^*(T)]^{W_f} \to 0$$

Localize the above sequence at $\mathfrak{m}$ and since this is exact we have $\mathcal{Z}(\mathcal{H})_\mathfrak{m}/\mathfrak{m}\mathcal{Z}(\mathcal{H})_\mathfrak{m} \cong \mathbb{Z}[X^*(T)]^{W_f}$. But since $sp(z_\lambda) = c_\lambda$, if we let $Z' = \bigoplus_{\lambda \in X^*_+(T)} \mathbb{Z}[q^{\pm 1}] \cdot z_\lambda$ we also have

$$0 \to \mathfrak{m}Z' \to Z' \xrightarrow{sp} \mathbb{Z}[X^*(T)]^{W_f} \to 0$$

and thus localizing at $\mathfrak{m}$ it follows that

$$\mathcal{Z}(\mathcal{H})_\mathfrak{m}/\mathfrak{m}\mathcal{Z}(\mathcal{H})_\mathfrak{m} \cong Z'_\mathfrak{m}/\mathfrak{m}Z'_\mathfrak{m} \tag{2}$$

$\mathcal{H}$ if f.g. ($\leq |W_f|$) over $\mathbb{Z}[X^*(T)]$) which itself is f.g. ($\leq |W_f|$) over Z'. Since Z' is Noetherian a ring it follows that $\mathcal{H}$ is a Noetherian module over Z' and $\mathcal{Z}(\mathcal{H})$, being a submodule, is thus a f.g. module over Z'. Since $Z'_\mathfrak{m}$ is a local ring and Eq. (2) shows that $\mathcal{Z}(\mathcal{H})_\mathfrak{m} = Z'_\mathfrak{m} + \mathfrak{m}\mathcal{Z}(\mathcal{H})_\mathfrak{m}$, by Nakayama's lemma it follows that $Z'_\mathfrak{m} = \mathcal{Z}(\mathcal{H})_\mathfrak{m}$.

Thus every $z \in \mathcal{Z}(\mathcal{H})$ can be written as a $\mathbb{Z}[q^{\pm 1}]_\mathfrak{m}$ sum of z_λ's and in fact must be a $\mathbb{Z}[q^{\pm 1}]$ sum as z, z_λ have no poles. Finally the z_λ clearly independent over $\mathbb{Z}[q^{\pm 1}]$ ■

Corollary 2. *All irreducibles of $\mathcal{H}$ are f.d.*

Proof. The above proof shows that $\mathcal{H}$ is finite rank ($\leq |W_f|^2$) over its center and thus follows from "big center trick." ■

2. The $\widetilde{A_1}$ case

Let $PGL_2 = (\mathbb{Z}, \{\pm 1\}, \mathbb{Z}, \{\pm 2\})$. Because $X^*(T) = \mathbb{Z}R$[1], $\mathcal{H} = \mathcal{H}(PGL_2) = \mathcal{H}_{IM}(\langle s, t\rangle) = \mathbb{C}\langle T_s, T_\sigma\rangle / \sim$. In particular, we have the quadratic relation $(T_s + 1)(T_s - q) = 0$. Thus, there are two obvious one dimensional irreducibles

$$\mathbb{C}_{triv} : T_s, T_\sigma \curvearrowright q \qquad \mathbb{C}_{\mathrm{St}} : T_s, T_\sigma \curvearrowright -1$$

In the Bernstein presentation we have $\mathcal{H}(PGL_2) = \mathbb{C}\left\langle T_s, \theta, \theta^{-1}\right\rangle / \sim$ and recall from Che's talk that $\theta = q^{-1}T_\sigma T_s$ and thus

$$\mathbb{C}_{triv} : \theta \curvearrowright q \qquad \mathbb{C}_{\mathrm{St}} : \theta \curvearrowright q^{-1}$$

Theorem 3. *(a) $L_t := \mathrm{Ind}^{\mathcal{H}}_{\mathbb{C}[\theta^{\pm 1}]}\mathbb{C}_t$ is irreducible $\forall t \in \mathbb{C}^\times \setminus \left\{q, q^{-1}, -1\right\}$.*

(b) $L_t \cong L_w \iff w = t^{-1}$.

(c) Besides $\{L_t\}$, the only other irreducibles of $\mathcal{H}(PGL_2)$ are $\mathbb{C}_{triv}, \mathbb{C}_{\mathrm{St}}$, and $\pi(-1, triv), \pi(-1, \mathrm{St})$. All are 1-dimensional and fit in SES

$$0 \to \mathrm{St} \to L_q \to triv \to 0$$
$$0 \to triv \to L_{q^{-1}} \to \mathrm{St} \to 0$$
$$0 \to \pi(-1, \mathrm{St}) \to L_{-1} \leftrightarrows \pi(-1, triv) \to 0$$

[1] Technically this should really be $\mathcal{H}(\mathrm{SL}_2)$ but Solleveld's notation is flipped so that $W_{aff} = W \ltimes R$.

Proof. (a) By Bernstein presentation we have $\mathcal{H} \overset{v.s}{=} \mathcal{H}_f \otimes \mathbb{C}[\theta^{\pm 1}]$ and thus $L_t \cong \mathcal{H}_f = \mathbb{C}(triv) \oplus \mathbb{C}(sgn)$ as $\mathcal{H}_f$-modules. Recall

$$\mathbb{C}(triv) = \mathbb{C}e_t, \; e_t = \frac{T_s + 1}{1+q}, \qquad \mathbb{C}(sgn) = \mathbb{C}e_s, \; e_s = \frac{T_s - q}{1+q}$$

Recall the defining commutation relation of $\mathcal{H}(PGL_2)$

$$\theta_1 T_s - T_s \theta_{-1} = (q-1)(\theta_1 + 1) \tag{3}$$

It follows that

$$\theta_1(T_s + 1) = T_s\theta_{-1} + q\theta_1 + (q-1) \tag{4}$$

If L_t is $\mathcal{H}-$reducible $\implies$ L_t is $\mathcal{H}_f-$reducible. and from above it must contain $\mathbb{C}(triv)$ or $\mathbb{C}(sgn)$ as an $\mathcal{H}_f$ submodule and since $\dim_{\mathbb{C}} L_t = 2$ it follows that $\mathbb{C}(triv)$ or $\mathbb{C}(sgn)$ must be a $\mathcal{H}$ submodule $\iff$

$$ke_t \overset{?}{=} \theta_1 e_t = \theta_1 \frac{(T_s+1)\otimes_\theta 1}{1+q} \overset{Eq.\ (4)}{=} \frac{(T_s t^{-1} + (qt + q - 1))\otimes_\theta 1}{1+q}$$

$\iff$ $qt + q - 1 = t^{-1}$ and using the quadratic formula we see that $t = q^{-1} or - 1$. A similar calculation with $\mathbb{C}(sgn)$ shows that $t = q$ or -1.

(b) Consider the element $f = \dfrac{\theta(q-1) + q - 1}{\theta - \theta^{-1}} \in \mathbb{C}(X^*(T))$. By direct computation we have

$$\theta(T_s - f) = (T_s - f)\theta^{-1} \tag{5}$$

For $t \notin \{1, -1\}$ f has well defined action on L_t and given $v \in \mathbb{C}_t \setminus \{0\}$ we have

$$\theta(T_s - f)(1 \otimes_\theta v) \overset{Eq.\ (5)}{=\!=} (T_s - f)\theta^{-1} \otimes_\theta v = t^{-1}(T_s - f)(1 \otimes_\theta v)$$

Since $1 \otimes_\theta 1$ generates the $t-$eigenspace it follows that $\mathrm{Res}_{\mathbb{C}[\theta]} L_t = \mathbb{C}_t \oplus \mathbb{C}_{t^{-1}}$. Now,

$$\begin{aligned} L_w \cong L_t \iff 0 &\neq \mathrm{Hom}_{\mathcal{H}}(\mathrm{Ind}^{\mathcal{H}}_{\mathbb{C}[\theta^{\pm 1}]}\mathbb{C}_w, L_t) = \mathrm{Hom}_{\mathbb{C}[\theta^{\pm 1}]}(\mathbb{C}_w, \mathrm{Res}_{\mathbb{C}[\theta^{\pm 1}]} L_t) \\ &= \mathrm{Hom}_{\mathbb{C}[\theta^{\pm 1}]}(\mathbb{C}_w, \mathbb{C}_t \oplus \mathbb{C}_{t^{-1}}) \\ &\iff w = t \text{ or } t^{-1} \end{aligned}$$

■

(c) We claim any irreducible π of $\mathcal{H}$ must be a quotient of L_t for some t. Since π is f.d. by Corollary 2 there must at at least one eigenvalue for θ say t. Now note

$$\mathrm{Hom}_{\mathcal{H}}(\mathrm{Ind}^{\mathcal{H}}_{\mathbb{C}[\theta^{\pm 1}]}\mathbb{C}_t, \pi) = \mathrm{Hom}_{\mathbb{C}[\theta^{\pm 1}]}(\mathbb{C}_t, \pi) \neq 0$$

which gives the surjection as π is irreducible.

Remark. The exact same argument in (c) shows that in general all simple $\mathcal{H}-$modules are quotients of $\mathrm{Ind}^{\mathcal{H}}_{\mathbb{C}[X^*(T)]}\mathbb{C}_t$ for some $t \in T$.

3. Local Langlands

Unramified Local Langlands: Let $\mathcal{K}$ be a non-Archimedean local field and $W_{\mathcal{K}}$ the Weil-Deligne group ("discrete" version of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$).

$$\begin{tikzcd}
\{\text{"spherical" reps of } G(\mathcal{K})\} \arrow[r, leftrightarrow, "1:1"] \arrow[d, leftrightarrow, "\wr"] & \{\text{"unramified" Weil-Deligne reps in } G^\vee(\mathbb{C})\} \arrow[d, leftrightarrow, "\wr"] \\
\{\text{reps of } G(\mathcal{K}) \text{ admitting a } G(\mathcal{O})-\text{fixed vector}\} \arrow[d, leftrightarrow, "\wr", "V\mapsto V^{G(\mathcal{O})}"'] & \{\text{semisimple elements in } G^\vee(\mathbb{C}) \text{ up to conjugacy}\} \\
\{\text{irr of } \mathcal{H}_{sph} = \mathrm{Fun}_c(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}),\mathbb{C})\} &
\end{tikzcd}$$

Theorem 2 (Satake)
$\mathcal{H}_{sph}$ is commutative and there is a canonical isomorphism

$$\mathcal{H}_{sph} \cong K_0(\mathrm{Rep}\, G^\vee)\otimes_{\mathbb{Z}}\mathbb{C}$$

By the Weyl character formula we know that $K_0(\mathrm{Rep} G^\vee) = \mathbb{Z}[X^*(T^\vee)]^{W_f}$ and thus

$$K_0(\mathrm{Rep}\, G^\vee)\otimes_{\mathbb{Z}}\mathbb{C} = \mathbb{C}[X^*(T^\vee)]^{W_f} = \mathcal{O}(T^\vee/W) = \mathcal{O}(G^\vee_{ss}/\text{conjugacy})$$

In type A, the last equality is true because semisimple elements in GL_n correspond to diagonalizable elements so up to conjugacy are parameterized by entries on diagonal modulo the order (action of S_n). Now we see that $\mathrm{Irr}\,\mathcal{H}_{sph} = \mathrm{Hom}(\mathcal{H}_{sph},\mathbb{C}) \overset{Satake}{=\!=} G^\vee_{ss}/\text{conjugacy}$ as desired.

Tamely ramified Local Langlands: Let $q = |\mathcal{O}/\mathfrak{m}|$ be the size of the residue field.

$$\begin{tikzcd}
\{\text{reps of } G(\mathcal{K}) \text{ admitting a } I-\text{fixed vector}\} \arrow[r, leftrightarrow, "finite:1"] \arrow[d, leftrightarrow, "\wr", "V\mapsto V^{I}"'] & \left\{\begin{array}{c}\text{"tamely ramified" Weil-Deligne reps in } G^\vee(\mathbb{C})\\ \text{with unipotent monodromy}\end{array}\right\} \arrow[d, leftrightarrow, "\wr"] \\
\{\text{irr of } \mathcal{H}_{ext} = \mathcal{H}(G(\mathcal{K}), I)\} & \left\{\text{ pairs } (s,N)\in G^\vee_{ss}(\mathbb{C})\times\mathcal{N}^\vee \,|sNs^{-1} = qN\right\}
\end{tikzcd}$$

where $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$.